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Instabilities and loss of ellipticity in fiber-reinforced compressible non-linearly elastic solids under plane deformation

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Abstract

In a recent paper we examined the loss of ellipticity and its interpretation in terms of fiber kinking and other instability phenomena in respect of a fiber-reinforced *incompressible* elastic material. Here we provide a corresponding analysis for fiber-reinforced *compressible* elastic materials. The analysis concerns a material model which consists of an isotropic base material augmented by a reinforcement dependent on the fiber direction. The assessment of loss of ellipticity can be cast in terms of the eigenvalues of the acoustic tensors associated with the isotropic and anisotropic parts of the strain-energy function. For the anisotropic part, two different reinforcing models are examined and it is shown that, depending on the choice of model and whether the fiber is under compression or extension, loss of ellipticity may be associated with, in particular, a weak surface of discontinuity normal to or parallel to the deformed fiber direction or at an intermediate angle. Under compression the associated failure interpretations include fiber kinking and fiber splitting, while under extension fiber de-bonding and matrix failure are included.

Keywords: Fiber failure; Fiber kinking; Fiber de-bonding; Fiber splitting; Matrix failure; Loss of ellipticity; Reinforcing models; Anisotropy

1. Introduction

In this paper we continue the study, initiated by Merodio and Ogden (2002), concerned with the analysis of failure mechanisms in transversely isotropic elastic solids. These failure mechanisms include fiber kinking (Budiansky and Fleck, 1993; Kyriakides et al., 1995; Vogler and Kyriakides, 1997; Jensen and Christoffersen, 1997; Budiansky et al., 1998; Moran and Shih, 1998; Merodio and Pence, 2001a,b), fiber splitting (Lee et al., 2000), fiber de-bonding (Piggott, 1997) and matrix failure (Okabe et al., 1999; Liao and

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Reifsnider, 2000). For a review of compressive failure mechanisms we refer to Fleck (1997), and for some recent experimental results and analysis of kink band propagation to Hsu et al. (1999) and Vogler and Kyriakides (1999). Our purpose here is to investigate further the reinforcement model presented by Merodio and Ogden (2002) with a view to interpreting the above-mentioned failure mechanisms in terms of loss of ellipticity of the governing partial differential equations (equivalently of the material model). A unified approach to the prediction of fiber instability or fiber failure in fiber-reinforced composite materials on the basis of loss of ellipticity has been provided recently by Merodio and Ogden (2002) in respect of *incompressible* elastic materials. In this paper we present a corresponding analysis for *compressible* elastic materials.

For a given strain-energy function the equation that defines the loss of ellipticity determines both the *deformation* associated with the existence of surfaces of weak or strong discontinuity and the *direction of the normal* to that surface. Surfaces of weak discontinuity (or weak surfaces) are surfaces across which the second derivative of the deformation field is discontinuous. A strong (or fully developed) surface of discontinuity, on the other hand, is one across which the first derivative (i.e. the deformation gradient) suffers a finite jump. In each case the emergence of such a surface is associated with the loss of ellipticity, and loss of ellipticity therefore heralds either a weak or strong discontinuity (or both). A strong surface is also necessarily a weak surface, and in our discussion we therefore refer for the most part to weak surfaces, it being understood that in some cases such a surface may also be strong.

In Merodio and Ogden (2002) we related the angle between the weak surface normal and the fiber-reinforcement direction to a particular failure mechanism. For example, the onset of fiber kinking under fiber compression was associated with a weak surface that lies close to the normal to the direction of fiber reinforcement (Budiansky and Fleck, 1993). For a weak surface whose normal lies close to the direction of fiber reinforcement, on the other hand, the failure mechanism may be interpreted as fiber de-bonding (Piggott, 1997). For fiber kinking combined with fiber splitting, the combination of weak surfaces close to and normal to the fiber direction is required (Lee et al., 2000). Matrix failure arises under fiber *extension* and is associated with weak surfaces whose normal is parallel to the fiber reinforcement (Okabe et al., 1999; Liao and Reifsnider, 2000). For further discussion we refer to Merodio and Ogden (2002).

Constitutive equations that suffer a loss of ellipticity have been studied in a variety of contexts (see, e.g., Knowles and Sternberg, 1975, 1978; Abeyaratne, 1980; Triantafyllidis and Abeyaratne, 1983; Zee and Sternberg, 1983; Rosakis, 1990; Horgan, 1996; Jensen and Christoffersen, 1997; Qiu and Pence, 1997; Merodio and Pence, 2001a,b). In particular, the loss of ellipticity of some particular *transversely isotropic* non-linearly elastic materials under *plane* deformations has been examined by Triantafyllidis and Abeyaratne (1983), Qiu and Pence (1997) and Merodio and Pence (2001a,b). In each of these latter works a constitutive model consisting of an isotropic base material augmented by a uniaxial reinforcement was used, the direction of reinforcement being referred to as the *fiber direction*. The fiber reinforcement is taken to lie in the considered plane of deformation in each case. Here, we adopt the same structure for the material model and define the strain energy as an isotropic base material augmented by a reinforcing model. For the latter, two general classes of functions are examined. We mention here that some related aspects of strong ellipticity in three-dimensions for transversely isotropic elastic solids have also been studied recently (Wilber and Walton, 2002; Walton and Wilber, 2003).

In three dimensions, two independent deformation invariants, denoted I_4 and I_5, are sufficient to characterize the anisotropic nature of a transversely isotropic material. These are additional to the usual three invariants I_1, I_2, I_3 of the Cauchy–Green deformation tensors required for isotropy in a compressible material. The notation is defined in Section 2.1. Under plane deformations only two of I_1, I_2, I_3 are independent and, if the fiber direction is taken to be in the considered plane, I_4 and I_5 are no longer independent but are connected through I_1 and I_3. The (in-plane) material response may then be regarded as depending only on I_1, I_2 and I_4 or, equivalently, on I_1, I_3 and I_4. The ellipticity status of a strain-energy function under the restriction to the plane in question then depends on only one anisotropic invariant. In the reinforcement model we shall consider separately dependence on I_4 and on an invariant that couples I_4 with I_1 since each model adds a distinct anisotropic character to the isotropic base material.

In Section 2, the material model is introduced and the ellipticity, strong ellipticity and loss of ellipticity conditions for the governing differential equations are summarized. Specialization to plane strain is discussed in Section 3. In Section 3.1, the ellipticity status of a general reinforcing model depending on I_4 is established. It is shown that failure of ellipticity is to be expected in fiber compression, and the incipient loss of ellipticity is interpreted in terms of fiber kinking. Failure can also occur in fiber extension if the reinforcing model loses convexity, with the weak surface (close to) parallel to the fiber direction, in which case fiber de-bonding is an appropriate interpretation of the associated failure mode. Convex reinforcing models are discussed briefly. The general results are then discussed in more detail in respect of a simple model for the isotropic base material (but without specializing the reinforcing model). This enables the discussion of loss of ellipticity to be conducted in terms of the eigenvalues of the acoustic tensors associated with the isotropic base material and the reinforcing model.

In Section 3.2, the analysis focuses on a reinforcing model for which I_4 and I_1 are coupled. Under fiber contraction it is found that failure of ellipticity may occur in two different modes, which may be associated with fiber kinking and fiber splitting. In fiber extension de-bonding is again a possible failure mode if the reinforcing model is non-convex. A weak surface may also arise perpendicular to the fiber direction and this is interpreted as matrix failure. Again we consider the specialization of the isotropic base material and continue the discussion briefly in terms of the eigenvalues of the acoustic tensors.

The examples of failure modes discussed here are not exhaustive. Other possibilities associated with loss of ellipticity may arise. For example, if the base material loses ellipticity then fiber de-bonding and matrix failure are also possible failure modes under fiber extension whether or not the reinforcing model is convex. Furthermore, other instabilities, not associated with loss of ellipticity, are possible, but are not considered here. In Section 4 we summarize and discuss briefly the results obtained in the previous sections.

2. Basic equations

2.1. Kinematics

Let $\mathbf{X}$, with Cartesian components X_α, $\alpha \in \{1,2,3\}$, denote the position vector of a material particle in the stress-free reference configuration and $\mathbf{x}$, with components x_i, $i \in \{1,2,3\}$, denote the corresponding position vector in the deformed configuration. The deformation gradient tensor is denoted $\mathbf{F}$ and has components $\partial x_i/\partial X_\alpha$. The left and right Cauchy–Green deformation tensors, respectively $\mathbf{B}$ and $\mathbf{C}$, are given by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \qquad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} \tag{1}$$

and the principal (isotropic) invariants of $\mathbf{C}$ (equivalently of $\mathbf{B}$) are defined by

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = I_3\,\mathrm{tr}\,(\mathbf{C}^{-1}), \quad I_3 = \det\mathbf{C}. \tag{2}$$

We consider a fiber-reinforcement defined in the undeformed configuration by the unit vector $\mathbf{A}$, which may depend on $\mathbf{X}$. The combination of $\mathbf{A}$ and $\mathbf{C}$ introduces two additional invariants, denoted I_4 and I_5, which are defined by

$$I_4 = \mathbf{A}\cdot(\mathbf{C}\mathbf{A}), \quad I_5 = \mathbf{A}\cdot(\mathbf{C}^2\mathbf{A}). \tag{3}$$

We use the notation $\mathbf{a}$ defined by

$$\mathbf{a} = \mathbf{F}\mathbf{A}. \tag{4}$$

Then, from (4), (1) and (3) we have

$$I_4 = \mathbf{a} \cdot \mathbf{a}, \quad I_5 = \mathbf{a} \cdot (\mathbf{Ba}). \tag{5}$$

In terms of the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$ of the deformation we have

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = I_3(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}), \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{6}$$

$$I_4 = \lambda_1^2 A_1^2 + \lambda_2^2 A_2^2 + \lambda_3^2 A_3^2 = a_1^2 + a_2^2 + a_3^2, \tag{7}$$

$$I_5 = \lambda_1^4 A_1^2 + \lambda_2^4 A_2^2 + \lambda_3^4 A_3^2 = \lambda_1^2 a_1^2 + \lambda_2^2 a_2^2 + \lambda_3^2 a_3^2, \tag{8}$$

where (A_1, A_2, A_3) are the components of $\mathbf{A}$ referred to the principal axes of $\mathbf{C}$ and (a_1, a_2, a_3) those of $\mathbf{a}$ referred to the principal axes of $\mathbf{B}$. It is clear from the above that I_4 is the square of the stretch in the direction $\mathbf{A}$ of the fiber reinforcement and therefore registers deformations that modify the length of the fiber. There is no corresponding immediate simple interpretation of I_5 in general. However, it is worth noting that if $\mathbf{A}$ is taken to correspond to the x_1 coordinate direction then $I_4 = C_{11}$, while, in general, $I_5 = I_4^2 + C_{12}^2 + C_{13}^2$ and hence I_5 also registers shearing deformations through the shear components C_{12} and C_{13} except when $\mathbf{A}$ is an eigenvector of $\mathbf{C}$ (and $C_{12} = C_{13} = 0$).

As discussed by Merodio and Ogden (2002) a connection between I_5 and the deformation of area elements may be established by use of the Cayley–Hamilton theorem for $\mathbf{C}$. This gives

$$I_5 = I_1 I_4 - I_2 + \mathbf{A} \cdot (\mathbf{C}^* \mathbf{A}), \tag{9}$$

where $\mathbf{C}^* = I_3 \mathbf{C}^{-1}$ is the adjugate of $\mathbf{C}$. Since a reference surface area element of unit magnitude with normal in the direction $\mathbf{A}$ transforms to $\sqrt{I_3}\mathbf{F}^{-\mathrm{T}}\mathbf{A}$ (Nanson's formula) the final term in (9) is interpreted as the square of the ratio of deformed to undeformed surface area elements.

2.2. Strain energy and stress

For an elastic material without internal constraints the most general strain-energy function for a homogeneous transversely isotropic non-linear elastic solid depends only on the invariants $(I_1, I_2, I_3, I_4, I_5)$. For details we refer to the work of Spencer (1972). Accordingly, we write the strain energy per unit reference volume as

$$W = W(I_1, I_2, I_3, I_4, I_5). \tag{10}$$

The nominal stress tensor $\mathbf{S}$ is in general given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{11}$$

To make this explicit in respect of (10) we use the formulas

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1\mathbf{F}^{\mathrm{T}} - 2\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \tag{12}$$

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{A} \otimes \mathbf{FA}, \quad \frac{\partial I_5}{\partial \mathbf{F}} = 2(\mathbf{A} \otimes \mathbf{FCA} + \mathbf{CA} \otimes \mathbf{FA}) \tag{13}$$

to obtain

$$\mathbf{S} = 2W_1\mathbf{F}^{\mathrm{T}} + 2W_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} + 2I_3W_3\mathbf{F}^{-1} + 2W_4\mathbf{A} \otimes \mathbf{FA} + 2W_5(\mathbf{A} \otimes \mathbf{FCA} + \mathbf{CA} \otimes \mathbf{FA}), \tag{14}$$

where the subscripts $1, \ldots, 5$ on W indicate differentiation with respect to $I_1, \ldots, I_5$, respectively, and $\mathbf{I}$ is the identity tensor.

The associated Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$J\boldsymbol{\sigma} = \mathbf{FS} = 2W_1\mathbf{B} + 2W_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2I_3W_3\mathbf{I} + 2W_4\mathbf{a}\otimes\mathbf{a} + 2W_5(\mathbf{a}\otimes\mathbf{Ba} + \mathbf{Ba}\otimes\mathbf{a}), \tag{15}$$

where $J = \det\mathbf{F} = I_3^{1/2}$.

The energy function and the stress must vanish in the reference configuration (where $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = 1$). Hence,

$$W(3,3,1,1,1) = 0, \tag{16}$$

$$W_1(3,3,1,1,1) + 2W_2(3,3,1,1,1) + W_3(3,3,1,1,1) = 0, \tag{17}$$

$$W_4(3,3,1,1,1) + 2W_5(3,3,1,1,1) = 0. \tag{18}$$

Moreover, for consistency with the classical linear theory of transversely isotropic elasticity the conditions

$$W_{11} + 4W_{12} + 4W_{22} + 2W_{13} + 4W_{23} + W_{33} = c_{11}/4, \tag{19}$$

$$W_2 + W_3 = (c_{12} - c_{11})/4, \quad W_1 + W_2 + W_5 = c_{44}/2, \tag{20}$$

$$W_{14} + 2W_{24} + 2W_{15} + W_{34} + 4W_{25} + 2W_{35} = (c_{13} - c_{12})/4, \tag{21}$$

$$W_{44} + 4W_{45} + 4W_{55} + 2W_5 = (c_{33} - c_{11} + 2c_{12} - 2c_{13})/4 \tag{22}$$

must be satisfied, where the derivatives of W are evaluated in the reference configuration and the constants $c_{11},\ldots,c_{44}$ constitute the standard notation for the elastic constants used in classical transverse isotropy with the x_3 coordinate direction corresponding to the axis of symmetry (see, e.g., Love, 1944, p. 160).

2.3. Equilibrium and ellipticity

The equation of equilibrium in the absence of body forces has the form $\operatorname{Div}\mathbf{S} = \mathbf{0}$ and may be written in the component form

$$\mathscr{A}_{\alpha i\beta j}x_{j,\alpha\beta} = 0, \tag{23}$$

where

$$\mathscr{A}_{\alpha i\beta j} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}, \tag{24}$$

Greek and Roman indices being associated with the reference and deformed configurations respectively. The subscripts following a comma indicate differentiation with respect to the relevant coordinate and the usual summation convention for repeated indices is adopted.

Consider a body whose reference configuration is denoted by $\mathscr{B}_r$. Suppose that $\mathbf{F}$ is continuous in $\mathscr{B}_r$ and that $\operatorname{Grad}\mathbf{F}$, with components $x_{i,\alpha\beta}$, is also continuous except that it may suffer a jump discontinuity across a surface $\mathscr{S}_r$ in $\mathscr{B}_r$. Such a surface is referred to as a *weak discontinuity*. Let $\mathbf{N}$ denote a normal vector to $\mathscr{S}_r$. Then, the jump in $x_{i,\alpha\beta}$ across $\mathscr{S}_r$ is given by

$$[x_{i,\alpha\beta}] = m_iN_\alpha N_\beta, \tag{25}$$

where $[\cdot]$ denotes the difference in the enclosed quantity evaluated on the two sides of $\mathscr{S}_r$ and m_i are the components of an undetermined vector $\mathbf{m}$.

By taking the difference of the left-hand side of (23) on the two sides of $\mathscr{S}_r$ and using (25) we obtain

$$\mathscr{A}_{\alpha i\beta j}m_iN_\alpha N_\beta = 0. \tag{26}$$

We now update (i.e. push forward) $\mathscr{A}_{\alpha i\beta j}$ to the deformed configuration to define $\mathscr{A}_{0piqj}$ by

$$J\mathscr{A}_{0piqj} = F_{p\alpha}F_{q\beta}\mathscr{A}_{\alpha i\beta j} \tag{27}$$

(see, e.g., Ogden, 1984; Holzapfel, 2000). We also define the vector **n** by

$$\mathbf{N} = \mathbf{F}^{\mathrm{T}}\mathbf{n}, \tag{28}$$

so that, recalling Nanson's formula $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-\mathrm{T}}\mathbf{N}\,\mathrm{d}A$ for the deformation of area elements, we see that **n** is normal to the image of $\mathscr{S}_r$ in the deformed configuration. We point out that while in Nanson's formula **N** and **n** are unit normals the **n** defined by (28) is not in general a unit vector.

Next, we use (27) and (28) to write (26) in the form

$$\mathbf{Q}(\mathbf{n})\mathbf{m} = \mathbf{0}, \tag{29}$$

where the *acoustic tensor* $\mathbf{Q}(\mathbf{n})$ has components defined by

$$Q_{ij} = \mathscr{A}_{0piqj}n_p n_q. \tag{30}$$

Thus, for the existence of a surface of weak discontinuity, $\mathbf{Q}(\mathbf{n})$ must be singular for some non-zero **n** and the associated value(s) of **n** is (are) given by

$$\det\mathbf{Q}(\mathbf{n}) = 0. \tag{31}$$

Once **n** is determined from (31) **m** is found from (29). It follows that for a deformation with the considered properties to be admissible the equality

$$\mathscr{A}_{0piqj}n_p n_q m_i m_j \equiv [\mathbf{Q}(\mathbf{n})\mathbf{m}]\cdot\mathbf{m} = 0 \tag{32}$$

must hold for some pair of non-zero vectors **m** and **n**. For a non-trivial solution this equation, together with (29), defines a pair of non-zero vectors **m** and **n**, the former being a null eigenvector of $\mathbf{Q}(\mathbf{n})$ for an **n** for which $\mathbf{Q}(\mathbf{n})$ is singular.

If the system of Eq. (23) is *elliptic* then no such solutions exist. The condition for ellipticity is that

$$[\mathbf{Q}(\mathbf{n})\mathbf{m}]\cdot\mathbf{m} \equiv \mathscr{A}_{0piqj}n_p n_q m_i m_j \neq 0 \tag{33}$$

for all vectors $\mathbf{m} \neq \mathbf{0}, \mathbf{n} \neq \mathbf{0}$. A stronger requirement is the *strong-ellipticity condition*

$$[\mathbf{Q}(\mathbf{n})\mathbf{m}]\cdot\mathbf{m} \equiv \mathscr{A}_{0piqj}n_p n_q m_i m_j > 0 \quad \mathbf{m} \neq 0, \;\; \mathbf{n} \neq 0. \tag{34}$$

Hence, for strong ellipticity $\mathbf{Q}(\mathbf{n})$ is positive definite for all vectors $\mathbf{n} \neq \mathbf{0}$. Without loss of generality we make take **m** and **n** to be unit vectors.

The analysis of Eq. (33) for specific forms of the energy function W furnishes the ellipticity status of that particular strain energy. A deformation gradient **F** satisfying (33) for every pair of unit vectors **m** and **n** is said to be an *elliptic deformation* for that W. If all possible deformations for a particular W are elliptic then the material itself is referred as an *elliptic material*. As is well known, the incompressible isotropic neo-Hookean material is an example of an elliptic material (see, e.g., Merodio and Ogden, 2002). On the other hand, if, for some pair of unit vectors **m** and **n**, a deformation gradient **F** satisfies equation (32), then the deformation is said to be non-elliptic for that material model. Furthermore, the vector **n** is identified as a normal vector to a weak surface (in the deformed configuration) as defined by the discontinuity (25). If **n** is taken to be a unit vector then, by (28), it follows that **N** is not in general a unit vector.

We emphasize here that a surface of strong discontinuity (across which **F** is discontinuous) is also defined by the loss of ellipticity condition (32), and, since discontinuity of **F** implies that its derivatives are also discontinuous, a strong surface is also a weak surface.

2.4. The reinforcing model

We consider an isotropic elastic material with strain energy denoted by $W_{\text{iso}}(I_1, I_2, I_3)$. This strain energy is then augmented to give a strain-energy function

$$W = W(I_1, I_2, I_3, I_4, I_5) = W_{\text{iso}}(I_1, I_2, I_3) + W_{\text{fib}}(I_4, I_5), \tag{35}$$

where the additional term $W_{\text{fib}}(I_4, I_5)$ represents the contribution due to the fiber reinforcement. The first term in (35) represents the *isotropic base material*, while the second term is the so-called *reinforcing model*, the subscript standing for 'fiber' reinforcement. This strain energy must be consistent with the conditions (16)–(22).

Henceforth, we restrict W_{fib} to functions that depend only on one invariant. First, we consider I_4 reinforcement and it will be convenient to write $W_{\text{fib}}(I_4, I_5) = F(I_4)$.

In the literature (see Triantafyllidis and Abeyaratne, 1983; Qiu and Pence, 1997) use has been made of the so-called *standard reinforcing model* defined by the function

$$F(I_4) = \tfrac{1}{2}\alpha(I_4 - 1)^2, \tag{36}$$

where $\alpha > 0$ is an anisotropy parameter measuring the degree of anisotropy or strength of reinforcement. (Note that a factor $1/2$ has been inserted in (36) compared with the definition used by Merodio and Ogden (2002).) The standard reinforcing model penalizes deformation in the fiber direction and is a convex function of I_4. In Triantafyllidis and Abeyaratne (1983) and Qiu and Pence (1997), for α sufficiently large, loss of ellipticity was found in fiber compression, i.e. for $I_4 < 1$. On the other hand, the considered materials were found to gain stability in fiber extension. These results were generalized in the paper by Merodio and Ogden (2002), which provided necessary and sufficient conditions for the ellipticity status of a general $F(I_4)$.

As discussed by Merodio and Ogden (2002) we note that the contribution of the term W_4 to the Cauchy stress (15) gives a traction component $2I_4W_4$ in the deformed fiber direction. Thus, for the reinforcing model $F(I_4)$ this contribution is positive (negative) in fiber extension (contraction) provided

$$F'(I_4) > 0 \ (< 0) \quad \text{for } I_4 > 1 \ (< 1), \qquad F'(1) = 0. \tag{37}$$

It may also be appropriate to take

$$F'(I_4) \to -\infty(\infty) \quad \text{as } I_4 \to 0(\infty), \tag{38}$$

although we note that the standard model (36) does not satisfy the lower of these limits.

In Merodio and Ogden (2002) a reinforcing model of the form $W_{\text{fib}}(I_4, I_5) = G(I_5)$ was also considered. The analogue of (37) is

$$G'(I_5) > 0 \ (< 0) \quad \text{for } I_5 > 1 \ (< 1), \qquad G'(1) = 0 \tag{39}$$

and of (38)

$$G'(I_5) \to -\infty(\infty) \quad \text{as } I_5 \to 0(\infty). \tag{40}$$

Note, however, that $I_5 > 1$ does not in general correspond to fiber extension, although $I_4 > 1$ implies $I_5 > 1$ and $I_5 < 1$ implies $I_4 < 1$, as in the incompressible situation (Merodio and Ogden, 2002). In what follows we shall adopt the inequalities (37)–(40), although it will be convenient in Section 3.2 to use a slightly different invariant in place of I_5.

3. The plane strain problem

Henceforth, as in the incompressible problem (Merodio and Ogden, 2002), we restrict attention to plane strain deformations and examine the ellipticity of the material model introduced above, with the fiber-reinforcement lying in the considered plane. We aim to derive conditions on the reinforcing model that provide a qualitative understanding of the ellipticity status of the energy function (35).

Let the considered plane correspond to the (X_1, X_2) coordinate plane so that the deformation satisfies $x_3 = X_3$ with (x_1, x_2) independent of X_3. Then, $F_{13} = F_{23} = F_{31} = F_{32} = 0$ and $F_{33} = 1$, and the components of $\mathbf{C}$ satisfy $C_{13} = C_{23} = 0$ and $C_{33} = 1$. The out-of-plane principal stretch is now $\lambda_3 = 1$ and the invariants (6) reduce to

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1, \quad I_2 = \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2. \tag{41}$$

The fiber direction $\mathbf{A}$ lies in the (X_1, X_2) plane, and therefore, from (7) and (8),

$$I_4 = \lambda_1^2 A_1^2 + \lambda_2^2 A_2^2, \qquad I_5 = \lambda_1^4 A_1^2 + \lambda_2^4 A_2^2. \tag{42}$$

It then follows that

$$I_2 = I_1 + I_3 - 1, \qquad I_5 = (I_1 - 1)I_4 - I_3, \tag{43}$$

while the specialization of (9) leads to

$$\mathbf{A} \cdot (\mathbf{C}^*\mathbf{A}) = I_1 - I_4 - 1. \tag{44}$$

Thus, the strain-energy function $W(I_1, I_2, I_3, I_4, I_5)$ of a fiber-reinforced elastic material (i.e. a transversely isotropic elastic material), when restricted to plane strain, can be represented in terms of three independent invariants, and we write

$$\widehat{W}(I_1, I_3, I_4) = W(I_1, I_1 + I_3 - 1, I_3, I_4, (I_1 - 1)I_4 - I_3). \tag{45}$$

We now adjust the notation so that $\mathbf{F}$ denotes the in-plane restriction of the deformation gradient and $\mathbf{A}$ is the corresponding in-plane fiber direction. Then, on specializing $(12)_1$ and $(13)_1$, we obtain

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \quad \frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{A} \otimes \mathbf{F}\mathbf{A}. \tag{46}$$

The corresponding plane restriction of the nominal stress tensor is then given by

$$\mathbf{S} = 2\widehat{W}_1\mathbf{F}^{\mathrm{T}} + 2I_3\widehat{W}_3\mathbf{F}^{-1} + 2\widehat{W}_4\mathbf{A} \otimes \mathbf{a}, \tag{47}$$

where $\mathbf{a} = \mathbf{F}\mathbf{A}$. Note that the only non-zero out-of-plane component of nominal stress (namely, S_{33}) required to maintain plane strain has to be calculated from (14) and is not given by (47). The corresponding (in-plane) Cauchy stress $\boldsymbol{\sigma}$ is given by

$$J\boldsymbol{\sigma} = 2\widehat{W}_1\mathbf{B} + 2\widehat{W}_3\mathbf{I} + 2\widehat{W}_4\mathbf{a} \otimes \mathbf{a} \tag{48}$$

specializing (15).

Restrictions on $\widehat{W}$ in the reference configuration analogous to those given for W in (16) and (17) are

$$\widehat{W}(3,1,1) = 0, \quad \widehat{W}_1(3,1,1) + \widehat{W}_3(3,1,1) = 0, \quad \widehat{W}_4(3,1,1) = 0, \tag{49}$$

while the appropriate specializations of (19)–(22) are

$$\begin{aligned} &\widehat{W}_{11} + 2\widehat{W}_{13} + \widehat{W}_{33} = c_{11}/4, \qquad 2\widehat{W}_{14} + 2\widehat{W}_{34} + \widehat{W}_{44} = (c_{33} - c_{11})/4, \\ &\widehat{W}_{44} - 2\widehat{W}_3 = (c_{11} + c_{33} - 2c_{13})/4, \qquad \widehat{W}_1 = c_{44}/2, \end{aligned} \tag{50}$$

all the derivatives of $\widehat{W}$ being evaluated at $(3, 1, 1)$. Note that these do not involve c_{12}.

For the $\widehat{W}$ defined above the components $\mathscr{A}_{0piqj}$ are explicitly

$$J\mathscr{A}_{0piqj} = 4\widehat{W}_{11}B_{pi}B_{qj} + 4I_3\widehat{W}_{13}(B_{pi}\delta_{qj} + B_{qj}\delta_{pi}) + 4I_3^2\widehat{W}_{33}\delta_{pi}\delta_{qj} + 4I_3\widehat{W}_{34}(\delta_{pi}a_q a_j + \delta_{qj}a_p a_i) + 4\widehat{W}_{14}(B_{pi}a_q a_j + B_{qj}a_p a_i) + 4\widehat{W}_{44}a_p a_i a_q a_j + 2\widehat{W}_1\delta_{ij}B_{pq} + 2I_3\widehat{W}_3(2\delta_{pi}\delta_{qj} - \delta_{pj}\delta_{qi}) + 2\widehat{W}_4\delta_{ij}a_p a_q, \quad (51)$$

wherein δ_{ij} denotes the Kronecker delta. In (51) and henceforth indices take the values 1 and 2 only.

In terms of components of $\mathbf{n}$ referred to the principal axes of $\mathbf{B}$ the components of $\mathbf{Q}$ are written

$$Q_{ij} = 4\widehat{W}_{11}\lambda_i^2\lambda_j^2 n_i n_j + 4I_3\widehat{W}_{13}(\lambda_i^2 + \lambda_j^2)n_i n_j + 4I_3^2\widehat{W}_{33}n_i n_j + 4I_3\widehat{W}_{34}(\mathbf{n}\cdot\mathbf{a})(n_i a_j + n_j a_i) + 4\widehat{W}_{14}(\mathbf{n}\cdot\mathbf{a}) \times (\lambda_i^2 n_i a_j + \lambda_j^2 n_j a_i) + 4\widehat{W}_{44}(\mathbf{n}\cdot\mathbf{a})^2 a_i a_j + 2\widehat{W}_1\delta_{ij}(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) + 2I_3\widehat{W}_3 n_i n_j + 2\widehat{W}_4\delta_{ij}(\mathbf{n}\cdot\mathbf{a})^2 \quad (52)$$

and the strong ellipticity condition (34), specialized to two dimensions, then becomes

$$2\widehat{W}_{11}[\mathbf{m}\cdot(\mathbf{Bn})]^2 + 4I_3\widehat{W}_{13}[\mathbf{m}\cdot(\mathbf{Bn})](\mathbf{m}\cdot\mathbf{n}) + 2I_3^2\widehat{W}_{33}(\mathbf{m}\cdot\mathbf{n})^2 + [4I_3\widehat{W}_{34}(\mathbf{m}\cdot\mathbf{n}) + 4\widehat{W}_{14}[\mathbf{m}\cdot(\mathbf{Bn})] + 2\widehat{W}_{44}(\mathbf{m}\cdot\mathbf{a})(\mathbf{n}\cdot\mathbf{a})](\mathbf{m}\cdot\mathbf{a})(\mathbf{n}\cdot\mathbf{a}) + \widehat{W}_1\mathbf{n}\cdot(\mathbf{Bn}) + I_3\widehat{W}_3(\mathbf{m}\cdot\mathbf{n})^2 + \widehat{W}_4(\mathbf{n}\cdot\mathbf{a})^2 > 0 \quad (53)$$

for all (in-plane) unit vectors $\mathbf{m}$ and $\mathbf{n}$, with $\mathbf{a} = \mathbf{FA}$ also being an in-plane vector.

When (53) is evaluated in the reference configuration and the notation (50) used it is easy to show that necessary and sufficient conditions for (53) to hold are

$$c_{11} > 0, \quad c_{33} > 0, \quad c_{44} > 0 \quad (54)$$

together with

$$| c_{13} + c_{44} | < \sqrt{c_{11}c_{33}} + c_{44}. \quad (55)$$

To extend the above to three-dimensional strong ellipticity the additional inequality $c_{11} > c_{12}$ is required (see, e.g., Payton, 1983). A simple proof of the above necessary and sufficient conditions is given by Merodio and Ogden (in press).

For the special case of an *isotropic* material the inequality (53) reduces to

$$2\widehat{W}_{11}[\mathbf{m}\cdot(\mathbf{Bn})]^2 + 4I_3\widehat{W}_{13}[\mathbf{m}\cdot(\mathbf{Bn})](\mathbf{m}\cdot\mathbf{n}) + 2I_3^2\widehat{W}_{33}(\mathbf{m}\cdot\mathbf{n})^2 + \widehat{W}_1\mathbf{n}\cdot(\mathbf{Bn}) + I_3\widehat{W}_3(\mathbf{m}\cdot\mathbf{n})^2 > 0, \quad (56)$$

with $\widehat{W}$ now independent of I_4. Necessary and sufficient conditions for (56) to hold for all non-zero $\mathbf{m}$ and $\mathbf{n}$ are well known (see, e.g., Knowles and Sternberg, 1977; Hill, 1979; Dowaikh and Ogden, 1991; Ogden and Sotiropoulos, 1998). They can be expressed in several different but equivalent forms, but are perhaps most transparent in terms of the stretches, with the strain energy written as a symmetric function $\breve{W}(\lambda_1, \lambda_2, \lambda_3)$ of the principal stretches. Then, if $\breve{W}_i$ and $\breve{W}_{ij}$, with $i, j \in \{1, 2\}$, denote, respectively, the first and second derivatives of $\breve{W}$ with respect to the stretches, necessary and sufficient conditions for (56) are

$$\breve{W}_{11} > 0, \quad \breve{W}_{22} > 0, \quad \frac{\lambda_1\breve{W}_1 - \lambda_2\breve{W}_2}{\lambda_1^2 - \lambda_2^2} > 0, \quad (57)$$

$$\sqrt{\breve{W}_{11}\breve{W}_{22}} - \breve{W}_{12} + \frac{\breve{W}_1 + \breve{W}_2}{\lambda_1 + \lambda_2} > 0, \quad (58)$$

$$\sqrt{\breve{W}_{11}\breve{W}_{22}} + \breve{W}_{12} - \frac{\breve{W}_1 - \breve{W}_2}{\lambda_1 - \lambda_2} > 0, \quad (59)$$

the latter two being given in this form by Dowaikh and Ogden (1991). Note that these do not require that $\lambda_3 = 1$ (which has been assumed here).

Since we may obtain, for example,

$$\lambda_1^2 \check{W}_{11} = 4\lambda_1^4 \widehat{W}_{11} + 8\lambda_1^2 I_3 \widehat{W}_{13} + 4I_3^2 \widehat{W}_{33} + 2\lambda_1^2 \widehat{W}_1 + 2I_3 \widehat{W}_3 \tag{60}$$

and similar expressions for $\check{W}_{22}$ and $\check{W}_{12}$, the inequalities $(57)_{1,2}$, (58) and (59) are not expressible simply in terms of $\widehat{W}$. Note, however, that the third inequality in (57) is simply $\widehat{W}_1 > 0$.

It is worth noting in passing that *sufficient* conditions for (56) to hold are $\widehat{W}_1 > 0$ together with

$$\text{matrix} \quad \begin{pmatrix} \widehat{W}_{11} & I_3 \widehat{W}_{13} \\ I_3 \widehat{W}_{13} & I_3^{3/2} (I_3^{1/2} \widehat{W}_3)_3 \end{pmatrix} \quad \text{is positive semi-definite.} \tag{61}$$

Here we assume that the inequalities (57)–(59) hold. Thus, by continuity, strong ellipticity holds in some neighbourhood of the reference configuration, and on any path of deformation from the reference configuration strong ellipticity holds until a deformation is met at which strong ellipticity just fails. This happens (if at all) when a point is reached at which strict inequality > 0 in (53) is replaced by $\geqslant 0$ with equality holding for some non-zero $\mathbf{m}$ and $\mathbf{n}$.

We note that the inequality (53) is equivalent to

$$Q_{11}(\mathbf{n}) > 0, \qquad Q_{11}(\mathbf{n}) Q_{22}(\mathbf{n}) - [Q_{12}(\mathbf{n})]^2 > 0 \tag{62}$$

jointly for all unit vectors $\mathbf{n}$, where the components of $\mathbf{Q}(\mathbf{n})$ are given by (52). In general, the second inequality in (62) is a quartic in (n_1, n_2) and while necessary and sufficient conditions for a quartic to be positive can be written down they are rather complicated and the results are not very useful from the viewpoint of subsequent analysis. We therefore consider special forms of constitutive law in order to assess the failure of ellipticity.

3.1. I_4 reinforcement

With the restriction to plane strain we now consider the strain energy

$$\widehat{W}(I_1, I_3, I_4) = W_{\text{iso}}(I_1, I_3) + W_{\text{fib}}(I_4) \tag{63}$$

in which an isotropic base material with strain energy $W_{\text{iso}}(I_1, I_3)$ is augmented by the reinforcing model $W_{\text{fib}}(I_4)$. This is the plane strain specialization of (41) with I_5 omitted. For this separable form of energy, in which the dependence of $\widehat{W}$ on (I_1, I_3) and I_4 is decoupled, the strong ellipticity condition (53) reduces to

$$\begin{aligned} &2E_{11}[\mathbf{m} \cdot (\mathbf{Bn})]^2 + 4I_3 E_{13}[\mathbf{m} \cdot (\mathbf{Bn})](\mathbf{m} \cdot \mathbf{n}) + 2I_3^2 E_{33}(\mathbf{m} \cdot \mathbf{n})^2 + E_1 \mathbf{n} \cdot (\mathbf{Bn}) + I_3 E_3 (\mathbf{m} \cdot \mathbf{n})^2 \\ &\quad + (\mathbf{a} \cdot \mathbf{n})^2 [F' + 2(\mathbf{a} \cdot \mathbf{m})^2 F''] > 0, \end{aligned} \tag{64}$$

where, for convenience, we have introduced the notations

$$E(I_1, I_3) = W_{\text{iso}}(I_1, I_3), \quad F(I_4) = W_{\text{fib}}(I_4) \tag{65}$$

and a prime signifies differentiation with respect to I_4.

We now consider the contribution of the anisotropic part separately, noting that in (64) only the latter two terms depend on $\mathbf{a} = \mathbf{FA}$ and I_4. We assume that the isotropic base material is strongly elliptic so that the inequality (56) is satisfied for $\widehat{W} = E$.

3.1.1. The influence of $F(I_4)$

From (64) it is clear that since $\mathbf{n}$ may be chosen so that $\mathbf{a} \cdot \mathbf{n} = 0$ the ellipticity status of the model (63) depends on the sign of

$$F'(I_4) + 2(\mathbf{a} \cdot \mathbf{m})^2 F''(I_4). \tag{66}$$

Since we may choose $\mathbf{m}$ so that $\mathbf{a} \cdot \mathbf{m} = \mathbf{0}$ it is clear that for (66) to be non-negative it is necessary that $F'(I_4) \geqslant 0$. If also $F''(I_4) \geqslant 0$ then (66) is non-negative for all (m_1, m_2). If, on the other hand, $F''(I_4) < 0$ then, recalling $(5)_1$,

$$F'(I_4) + 2(\mathbf{a} \cdot \mathbf{m})^2 F''(I_4) \geqslant F'(I_4) + 2I_4 F''(I_4).$$

It follows that (66) is non-negative for all $\mathbf{m}$ if and only if

$$F'(I_4) \geqslant 0, \qquad F'(I_4) + 2I_4 F''(I_4) \geqslant 0. \tag{67}$$

Thus, sufficient conditions for (64) are clearly (67) together with (56) for $W_{\rm iso} = E$.

The factor $(\mathbf{n} \cdot \mathbf{a})^2$ in (64) ensures that, in isolation from the isotropic base material, $F(I_4)$ always loses ellipticity since $\mathbf{n}$ may be chosen so that $\mathbf{n} \cdot \mathbf{a} = 0$. For all other $\mathbf{n}$ the contribution of F to (64) is strictly positive if and only if

$$F'(I_4) > 0, \qquad F'(I_4) + 2I_4 F''(I_4) > 0. \tag{68}$$

Recalling (37) we see that the first of these inequalities fails in the reference configuration and in compression ($I_4 < 1$). The second holds in the reference configuration. For the standard reinforcing model (36) we note that $F'(I_4) + 2I_4 F''(I_4) > 0$ if and only if $I_4 > 1/3$. Deformation gradients $\mathbf{F}$ for which $I_4 < 1/3$ are not of interest since ellipticity will be lost at a larger value of $I_4 < 1$ on a path from $I_4 = 1$, as pointed out by Merodio and Ogden (2002).

3.1.2. Overall ellipticity

We are now concerned with the influence of the reinforcing model $F(I_4)$ on the overall ellipticity of the energy function (63). Without loss of generality we may take $F(1) = 0$. Hence, recalling (37), the restrictions on F in the reference configuration are

$$F(1) = 0, \quad F'(1) = 0, \quad E_{11}(3,1) + 2E_{13}(3,1) + E_{33}(3,1) + F''(1) > 0, \tag{69}$$

the latter following from $(54)_2$ with (50), c_{11} now dependent only on $E = W_{\rm iso}$. This is certainly satisfied if $F''(1) \geqslant 0$, which, in fact, follows from (37). Note, however, that (55), when combined with $(69)_3$, requires

$$2E_1(3,1) + F''(1) > [E_1(3,1)]^2 / (E_{11}(3,1) + 2E_{13}(3,1) + E_{33}(3,1)). \tag{70}$$

Since we assume that strong ellipticity holds in the reference configuration it follows by continuity that it also holds within some neighbourhood of the reference configuration in the space of deformation gradients $\mathbf{F}$. We denote this neighbourhood by E and refer to it as the *effective elliptic region* for $\widehat{W}$. The boundary of E, denoted $\partial\mathsf{E}$, is defined by the loss of strong ellipticity condition, i.e. by the set of deformation gradients for which

$$2E_{11}[\mathbf{m} \cdot (\mathbf{Bn})]^2 + 4I_3 E_{13}[\mathbf{m} \cdot (\mathbf{Bn})](\mathbf{m} \cdot \mathbf{n}) + 2I_3^2 E_{33}(\mathbf{m} \cdot \mathbf{n})^2 + E_1 \mathbf{n} \cdot (\mathbf{Bn}) + I_3 E_3 (\mathbf{m} \cdot \mathbf{n})^2$$
$$+ (\mathbf{a} \cdot \mathbf{n})^2 [F' + 2(\mathbf{a} \cdot \mathbf{m})^2 F''] \geqslant 0, \tag{71}$$

for all unit vectors $\mathbf{m}$ and $\mathbf{n}$, with equality holding for some pair of unit vectors $(\mathbf{m}, \mathbf{n})$, not necessarily unique. Since the isotropic base material is assumed to be strongly elliptic it is clear that a necessary condition for the breakdown of ellipticity of an elliptic isotropic non-linearly elastic solid augmented with the reinforcing model $F(I_4)$ is that, for $\mathbf{F} \in \mathsf{E}$, either $F'(I_4) < 0$ or $F'(I_4) + 2I_4 F''(I_4) < 0$ before the boundary $\partial\mathsf{E}$ is reached on any path of deformation from the reference configuration that crosses $\partial\mathsf{E}$.

We emphasize that a weak surface cannot be parallel to the fiber-reinforcement axis since then we would have $\mathbf{n} \cdot \mathbf{a} = 0$ and, because of the assumed strong ellipticity of the isotropic base material, the inequality (64) holds. Since weak surfaces are the only possible carriers of discontinuity for the equilibrium equations

(23), no surface of discontinuity, either weak or strong, can be aligned with the fiber direction. We recall that, for the standard reinforcing model, this result was established by Merodio and Pence (2001a) with a particular deformation on one side of the surface, namely a deformation for which the (reference) fiber direction is a Lagrangian principal direction (i.e. an eigenvector of $\mathbf{C}$). For an incompressible material the counterpart of the result given here was obtained by Merodio and Ogden (2002).

Let us now write $\mathbf{Q} = \mathbf{Q}^{\text{iso}} + \mathbf{Q}^{\text{fib}}$, where $\mathbf{Q}^{\text{iso}}$ is the acoustic tensor associated with $E(I_1, I_3)$ and $\mathbf{Q}^{\text{fib}}$ that with $F(I_4)$. Then, for the considered two-dimensional situation,

$$\det \mathbf{Q} = \det \mathbf{Q}^{\text{iso}} + Q_{11}^{\text{iso}} Q_{22}^{\text{fib}} + Q_{22}^{\text{iso}} Q_{11}^{\text{fib}} - 2Q_{12}^{\text{iso}} Q_{12}^{\text{fib}} + \det \mathbf{Q}^{\text{fib}}, \tag{72}$$

where, explicitly,

$$\det \mathbf{Q}^{\text{fib}} = 4(\mathbf{n} \cdot \mathbf{a})^4 F'(F' + 2I_4 F''). \tag{73}$$

In view of the assumed strong ellipticity of E we have

$$\det \mathbf{Q}^{\text{iso}} \equiv Q_{11}^{\text{iso}} Q_{22}^{\text{iso}} - Q_{12}^{\text{iso}^2} > 0, \quad Q_{11}^{\text{iso}} > 0, \quad Q_{22}^{\text{iso}} > 0. \tag{74}$$

Suppose, next, that we write $F(I_4) = \tilde{\alpha}\widetilde{F}(I_4)$, where $\tilde{\alpha}(> 0)$ is a dimensionless anisotropy parameter (compare the dimensional α in the standard reinforcing model). Then, Eq. (72) is expressed simply as a quadratic in $\tilde{\alpha}$ and we write this as

$$p(\tilde{\alpha}) = u + v\tilde{\alpha} + w\tilde{\alpha}^2, \tag{75}$$

where

$$u = \det \mathbf{Q}^{\text{iso}}, \quad v = Q_{11}^{\text{iso}} \widetilde{Q}_{22}^{\text{fib}} + Q_{22}^{\text{iso}} \widetilde{Q}_{11}^{\text{fib}} - 2Q_{12}^{\text{iso}} \widetilde{Q}_{12}^{\text{fib}}, \quad w = \det \widetilde{\mathbf{Q}}^{\text{fib}} \tag{76}$$

and $\widetilde{\mathbf{Q}}^{\text{fib}} = \mathbf{Q}^{\text{fib}}/\tilde{\alpha}$.

Clearly, if $\mathbf{n} \cdot \mathbf{a} \neq 0$ and $\det \mathbf{Q}^{\text{fib}} < 0$ then as the reinforcement strength is increased (i.e. as $\tilde{\alpha}$ is increased from zero) a value of $\tilde{\alpha}$ will be reached for which (72) vanishes for some unit vector $\mathbf{n}$, i.e. ellipticity is lost. This will certainly happen if $\tilde{\alpha}$ is sufficiently large but may not if the reinforcement is relatively weak.

To illustrate the possibilities we now consider separately the cases in which $F''(I_4) \geqslant 0$ and $F''(I_4) < 0$.

Case (a): $F''(I_4) \geqslant 0$. In this case a *necessary* condition for failure of ellipticity is $F'(I_4) < 0$, so that the fibers are under compressive strain. The largest negative contribution of

$$(\mathbf{a} \cdot \mathbf{n})^2 [F' + 2(\mathbf{a} \cdot \mathbf{m})^2 F''] \tag{77}$$

to (64) is

$$I_4 F'(I_4) \tag{78}$$

and it arises for $\mathbf{n} = \hat{\mathbf{a}}$ with $\mathbf{m} \cdot \mathbf{a} = 0$, where $\hat{\mathbf{a}} = \mathbf{a}/ \mid \mathbf{a} \mid$. This negative value increases (in magnitude) as I_4 decreases from unity and breakdown of ellipticity occurs when the negative value of $I_4 F'(I_4)$ balances the positive value of the terms in E in (64) with $\mathbf{n} = \hat{\mathbf{a}}$. For example, if $\hat{\mathbf{a}}$ is an eigenvector of $\mathbf{B}$ corresponding to the stretch λ_1 then for $\mathbf{m} \cdot \mathbf{n} = 0$ the left-hand side of (71) reduces to $I_4 F'(I_4) + \lambda_1^2 E_1(I_1, I_3)$. Since, by strong ellipticity of the base material, $E_1(I_1, I_3) > 0$, this will vanish for some $I_4 < 1$ even for reinforcements of moderate strength. For very strong reinforcement it will vanish for I_4 close to 1. For this pair of values of $\mathbf{n}$ and $\mathbf{m}$ it is easy to show that $[\mathbf{Q}^{\text{iso}}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m}$ is an eigenvalue of $\mathbf{Q}^{\text{iso}}(\mathbf{n})$ and hence an extremum corresponding to the most negative contribution of the F terms in (64). Since the weak surface is in this case normal to the fiber direction we may regard *fiber kinking* as the relevant failure mechanism under compressive strain in the fiber direction ($I_4 < 1$). Of course, fiber kinking is a strong discontinuity and in this case the weak surface is also a strong surface of discontinuity. However, in general, the extreme value of $[\mathbf{Q}^{\text{iso}}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m}$ may not be a minimum for the considered pair $\mathbf{m}$ and $\mathbf{n}$ and loss of ellipticity may therefore occur at an earlier point of the deformation path for an $\mathbf{n} \neq \hat{\mathbf{a}}$. This is discussed further in Section 3.1.4.

Case (b): $F''(I_4) < 0$. If $F''(I_4) < 0$ and $F'(I_4) + 2I_4F''(I_4) \geqslant 0$ then there can be no loss of ellipticity, but if $F'(I_4) + 2I_4F''(I_4) < 0$ then the (negative) least value of (77) is

$$I_4F' + 2I_4^2F'' \tag{79}$$

whether $F' > 0$ or $F' < 0$, and it occurs for $\mathbf{m} = \mathbf{n} = \hat{\mathbf{a}}$. In compression we necessarily have, by continuity, $F''(I_4) > 0$ for I_4 close to 1 so that in compression loss of ellipticity is likely to occur first in the mode discussed in (a) above. In fiber extension, on the other hand, since $F'(I_4) > 0$ loss of ellipticity occurs first, if at all, when $F'(I_4) + 2I_4F''(I_4)$ has passed from positive to negative. This, of course, requires loss of convexity of F.

Thus, for example, if we again suppose that $\mathbf{a}$ is an eigenvector of $\mathbf{B}$ then in fiber *extension* ellipticity can fail when $\mathbf{n}\cdot\mathbf{a}$ is small since the negative contribution to (77) then balances the positive contribution due to E provided the reinforcement is sufficiently strong. In this case the weak surface is close to parallel to the fiber direction and the relevant failure mechanism can be interpreted as de-bonding. Since strong ellipticity certainly holds when $\mathbf{n}\cdot\mathbf{a} = 0$ it might seem surprising that failure of ellipticity can occur for small $\mathbf{n}\cdot\mathbf{a}$. To illustrate why this can happen we consider the following example.

Let $\mathbf{a}$ be eigenvector of $\mathbf{B}$, corresponding to the stretch λ_1 say, and take $\mathbf{m} = \hat{\mathbf{a}}$. Then equality holds in (71) when

$$(2E_{11}\lambda_1^4 + 4I_3E_{13}\lambda_1^2 + 2I_3^2E_{33} + I_3E_3 + F' + 2I_4F'')n_1^2 + E_1(\lambda_1^2n_1^2 + \lambda_2^2n_2^2) = 0. \tag{80}$$

Again we write $F(I_4) = \tilde{\alpha}\widetilde{F}(I_4)$. Then, if $\tilde{\alpha}$ is sufficiently large then, on use of $n_2^2 = 1 - n_1^2$, (80) may be approximated as $(\mathbf{n}\cdot\hat{\mathbf{a}})^2 \equiv n_1^2 \approx E_1\lambda_2^2/[-\tilde{\alpha}(\widetilde{F}' + 2I_4\widetilde{F}'')]$, which is close to zero. Thus, in this case the weak surface is close to parallel to the fiber direction.

It is interesting to recall that, as discussed by Merodio and Ogden (2002), the contribution of $F(I_4)$ to the component of nominal traction, s say, in the fiber direction is, from (47), $2I_4^{1/2}F'(I_4)$. Hence, $\mathrm{d}s/\mathrm{d}I_4 = I_4^{-1/2}[F'(I_4) + 2I_4F''(I_4)]$ and failure of ellipticity therefore occurs after s has passed through a maximum during fiber extension.

We remark that, as discussed above, ellipticity cannot fail for an $\mathbf{n}$ satisfying $\mathbf{n}\cdot\mathbf{a} = 0$ if E is strongly elliptic. However, we record here that if E is allowed to lose ellipticity then this *can* happen for such an $\mathbf{n}$, i.e. when the weak surface coincides with the fiber direction. This, of course, is independent of the properties of the reinforcing model $F(I_4)$.

3.1.3. Consequences of $F''(I_4) \geqslant 0$

Here we examine briefly some implications of the convexity of $F(I_4)$, i.e. $F''(I_4) \geqslant 0$. Once more we write $F(I_4) = \tilde{\alpha}\widetilde{F}(I_4)$. Then loss of ellipticity requires fiber contraction since $F'(I_4) \geqslant 0$ and $F''(I_4) \geqslant 0$ in fiber extension. Furthermore, the breakdown of ellipticity for the considered materials, i.e. models with a strongly elliptic isotropic base material, satisfies a nesting property with respect to the parameter $\tilde{\alpha}$, as discussed by Merodio and Ogden (2002) for incompressible materials. The corresponding result in the present context is encapsulated in the following proposition.

Proposition. *If* $\mathbf{F}$ *is on the ellipticity boundary* $\partial\mathsf{E}$ *for* $\tilde{\alpha} = \tilde{\alpha}_1$ *and* $\tilde{\alpha}_2 > \tilde{\alpha}_1$ *then* $\mathbf{F} \notin \mathsf{E}$ *for* $\tilde{\alpha} = \tilde{\alpha}_2$.

Proof. This follows from (64), which we now write as

$$(\mathbf{Q}^{\mathrm{iso}}\mathbf{m})\cdot\mathbf{m} + \tilde{\alpha}(\widetilde{\mathbf{Q}}^{\mathrm{fib}}\mathbf{m})\cdot\mathbf{m} > 0. \tag{81}$$

By hypothesis $(\mathbf{Q}^{\mathrm{iso}}\mathbf{m})\cdot\mathbf{m} > 0$. Now suppose that the left-hand side of (81) vanishes for the deformation gradient $\mathbf{F}$ when $\tilde{\alpha} = \tilde{\alpha}_1$ and for a specific $\mathbf{n}$ (and hence $\mathbf{m}$) but is otherwise non-negative. It follows that for this $\mathbf{F}$ and the associated $\mathbf{m}$ and $\mathbf{n}$ the left-hand side of (81) is negative for $\tilde{\alpha}_2 > \tilde{\alpha}_1$. Hence, $\mathbf{F} \notin \mathsf{E}$ for $\tilde{\alpha} = \tilde{\alpha}_2$. □

In Qiu and Pence (1997) this nesting property of $\mathbf{F}$ and its connection with loss of ellipticity was illustrated for the standard reinforcing model (36). For an incompressible material with a general reinforcing model of the type considered here a corresponding Proposition was established by Merodio and Ogden (2002).

The situation for a non-elliptic isotropic base material was also discussed by Merodio and Ogden (2002) in respect of an incompressible material. There it was shown that if the fiber is under contraction then the same nesting property applies as for an elliptic base material, while if the fiber is subject to extension with a deformation gradient $\mathbf{F}$, then if $\mathbf{F} \in \mathsf{E}$ or $\partial\mathsf{E}$ for $\tilde{\alpha}_1$, then $\mathbf{F} \in \mathsf{E}$ is (strongly) elliptic for $\tilde{\alpha}_2 > \tilde{\alpha}_1$. Therefore, deformation gradients giving rise to breakdown of ellipticity are nested with respect to $\tilde{\alpha}$ in fiber contraction, while the elliptic regions are nested with respect to $\tilde{\alpha}$ in fiber extension. These results apply also for the compressible material considered here and we may conclude that an elliptic isotropic base material augmented with a convex reinforcing model gains stability in fiber extension while it is weakened in fiber contraction. Similarly, as $\tilde{\alpha}$ increases, i.e. as the degree of anisotropy increases, the solid becomes more stable in fiber extension, but less stable in fiber compression.

3.1.4. A specific material model

In order to highlight some of the features discussed above and to make the results more transparent we now consider the strain-energy function defined by

$$W(I_1, I_2, I_3, I_4, I_5) = \widehat{W}(I_1, I_3, I_4) = \mu(I_1 - 3) + H(I_3) + F(I_4), \tag{82}$$

where μ is a positive material constant, $H(I_3)$ is a function satisfying

$$H(1) = 0, \quad H'(1) = -\mu, \tag{83}$$

so that the conditions (49) are met, and $F(I_4)$ is subject to (37).

It is convenient to use the notation $\kappa = H''(1)$. Then, from (50), we obtain the specializations

$$c_{11} = 4\kappa, \quad c_{33} = 4(\kappa + \alpha), \quad c_{44} = 2\mu, \quad c_{13} = 4(\kappa - \mu), \tag{84}$$

where, as in (36), we have set $\alpha = F''(1)$. For strong ellipticity to hold in the reference configuration we obtain, from (54),

$$\mu > 0, \quad \kappa > 0, \quad \kappa + \alpha > 0. \tag{85}$$

The inequality (55) then requires the additional restriction $\alpha > -\mu(2\kappa - \mu)/\kappa$. In the present context, however, we have set $\alpha > 0$.

For the model (82) the components of $\mathbf{Q}^{\text{iso}}$ are obtained from (52) as

$$\mathbf{Q}^{\text{iso}} = \begin{pmatrix} (d + b_1)n_1^2 + b_2 n_2^2 & d n_1 n_2 \\ d n_1 n_2 & b_1 n_1^2 + (d + b_2)n_2^2 \end{pmatrix}, \tag{86}$$

where we have introduced the notations

$$d = 4I_3^2 H''(I_3) + 2I_2 H'(I_3), \quad b_1 = 2\mu\lambda_1^2, \quad b_2 = 2\mu\lambda_2^2. \tag{87}$$

Note that in the reference configuration we obtain $d = 4\kappa - 2\mu$, $b_1 = b_2 = 2\mu$ so that d may be positive or negative. In general, for strong ellipticity (positive definiteness of (86)) it is easy to deduce that necessary and sufficient conditions are simply $b_1 > 0$, $b_2 > 0$, $d + b_1 > 0$, $d + b_2 > 0$.

From (52) the components of $\mathbf{Q}^{\text{fib}}$ are read off as

$$\mathbf{Q}^{\text{fib}} = 2(\mathbf{n} \cdot \mathbf{a})^2 \begin{pmatrix} F' + 2a_1^2 F'' & 2F'' a_1 a_2 \\ 2F'' a_1 a_2 & F' + 2a_2^2 F'' \end{pmatrix}. \tag{88}$$

The eigenvalues of $\mathbf{Q}^{\text{iso}}$, denoted q_1^{iso} and q_2^{iso}, are easily found to be

$$q_1^{\text{iso}} = b_1 n_1^2 + b_2 n_2^2, \qquad q_2^{\text{iso}} = d + q_1^{\text{iso}} \tag{89}$$

and those of $\mathbf{Q}^{\mathrm{fib}}$, denoted q_1^{fib} and q_2^{fib}, are

$$q_1^{\mathrm{fib}} = 2(\mathbf{n}\cdot\mathbf{a})^2 F', \qquad q_2^{\mathrm{fib}} = 2(\mathbf{n}\cdot\mathbf{a})^2(F' + 2I_4F''). \tag{90}$$

For any given $\mathbf{n}$ the eigenvectors $\mathbf{m}$ associated with these eigenvalues are such that

$$q_1^{\mathrm{iso}} : \mathbf{m}\cdot\mathbf{n} = 0, \qquad q_2^{\mathrm{iso}} : \mathbf{m} = \mathbf{n}, \tag{91}$$

$$q_1^{\mathrm{fib}} : \mathbf{m}\cdot\mathbf{a} = 0, \qquad q_2^{\mathrm{fib}} : \mathbf{m} = \hat{\mathbf{a}}, \tag{92}$$

and we recall that $\hat{\mathbf{a}} = \mathbf{a}/\mid\mathbf{a}\mid$ and that $\mathbf{a}$ depends on the deformation through $\mathbf{a} = \mathbf{FA}$.

For a given $\mathbf{n}$ the eigenvalues $(q_1^{\mathrm{iso}}, q_2^{\mathrm{iso}})$ and $(q_1^{\mathrm{fib}}, q_2^{\mathrm{fib}})$ are extreme values of $(\mathbf{Q}^{\mathrm{iso}}\mathbf{m})\cdot\mathbf{m}$ and $(\mathbf{Q}^{\mathrm{fib}}\mathbf{m})\cdot\mathbf{m}$ respectively and it is therefore appropriate to consider if the least (positive) eigenvalue of $\mathbf{Q}^{\mathrm{iso}}$ can be associated with the least (negative) eigenvalue of $\mathbf{Q}^{\mathrm{fib}}$ at the point of loss of ellipticity. Now, it is clear from (91) and (92) that the combinations $(q_1^{\mathrm{iso}}, q_2^{\mathrm{fib}})$ and $(q_2^{\mathrm{iso}}, q_1^{\mathrm{fib}})$ are not admissible since they both entail $\mathbf{n}\cdot\mathbf{a} = 0$, in which case strong ellipticity holds. We therefore consider the combinations $(q_1^{\mathrm{iso}}, q_1^{\mathrm{fib}})$ and $(q_2^{\mathrm{iso}}, q_2^{\mathrm{fib}})$.

Case (i): $d > 0$. In this case $q_1^{\mathrm{iso}} < q_2^{\mathrm{iso}}$ and the least eigenvalue corresponds to $\mathbf{m}\cdot\mathbf{n} = 0$. We have

$$q_1^{\mathrm{iso}} + q_1^{\mathrm{fib}} = b_1 n_1^2 + b_2 n_2^2 + 2I_4F'. \tag{93}$$

This cannot vanish if $I_4 > 1$ but it can vanish for $I_4 < 1$ and the weak surface corresponding to loss of ellipticity is normal to the fibers ($\mathbf{n} = \hat{\mathbf{a}}$). For the standard reinforcing model this gives

$$1 - I_4 = \frac{\mu}{\alpha I_4}(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) \tag{94}$$

and for strong reinforcement ($\mu/\alpha \ll 1$) this yields a value of I_4 close to unity. As discussed earlier this is associated with fiber kinking.

Next, we consider

$$q_2^{\mathrm{iso}} + q_2^{\mathrm{fib}} = d + b_1 n_1^2 + b_2 n_2^2 + 2I_4(F' + 2I_4F''). \tag{95}$$

If $I_4 > 1$ this cannot vanish for $F'' > 0$ but it could (in principle) vanish if F'' passes from positive to negative as I_4 increases from 1. If so then this again corresponds to a weak surface normal to the fibers, but with $\mathbf{m} = \mathbf{n} = \hat{\mathbf{a}}$. This mode of loss of ellipticity might be associated with matrix failure. Note, however, that for the standard reinforcing model this possibility does not arise.

To be more specific we specialize the deformation to correspond to pure homogeneous strain with the principal axes of $\mathbf{B}$ corresponding to the Cartesian coordinate axes. Further, for simplicity of illustration, we take the deformation to be isochoric and write the stretches as $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$. Then $d = 4\kappa - 2\mu$. We also take the fiber direction to coincide with the x_1 axis, so that $I_4 = \lambda^2$, and F to correspond to the standard reinforcing model (36). From (72) we calculate $\det\mathbf{Q}$ and the resulting equation for this to vanish is written

$$\begin{aligned}&[\bar{\alpha}(\lambda^2 - 1) + 1][3\bar{\alpha}\lambda^4 - (\bar{\alpha} - 1)\lambda^2 + \delta]\lambda^6 n_1^4 + (\delta\lambda^2 + 1)n_2^4 + [3\bar{\alpha}\delta\lambda^6 - (\bar{\alpha} - 1)\delta\lambda^4 \\ &\quad + 4\bar{\alpha}\lambda^4 - 2(\bar{\alpha} - 1)\lambda^2 + \delta]\lambda^2 n_1^2 n_2^2 = 0,\end{aligned} \tag{96}$$

which involves two dimensionless material constants defined by $\bar{\alpha} = \alpha/\mu, \delta = d/2\mu$. For the considered material model there is no loss of ellipticity for $I_4 = \lambda^2 > 1$ so, in Fig. 1, we plot n_1^2 against $\lambda^2 < 1$ for three different values of the reinforcing parameter $\bar{\alpha}$ and one fixed value of δ (which is a measure of compressibility).

The results indicate that under compression loss of ellipticity occurs first for $n_1 = 1$ and that the stronger the reinforcement the closer to $\lambda = 1$ this happens. For the considered values of $\bar{\alpha}$ and δ there is only one

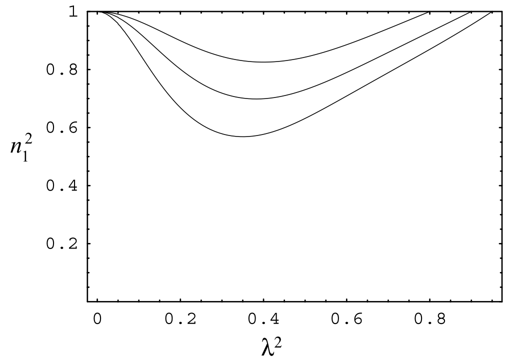

Fig. 1. Plot of the value of n_1^2 against λ^2 in compression corresponding to loss of ellipticity for the model (96) with dimensionless material parameters $\delta = 40$ and $\bar{\alpha} = 5, 10, 20$.

solution of Eq. (96), and this corresponds to $\lambda^2 = (\bar{\alpha} - 1)/\bar{\alpha}$ when $n_1 = 1$. A solution corresponding to vanishing of the coefficient $3\bar{\alpha}\lambda^4 - (\bar{\alpha} - 1)\lambda^2 + \delta$ of n_1^4 in (96) for $n_1 = 1$ requires a much smaller value of δ. Such solutions have the same character as the results shown in Fig. 1 but give a prior loss of ellipticity if and only if $\delta < -2(\bar{\alpha} - 1)^2/\bar{\alpha}$.

Case (ii): $d < 0$. In this case $q_2^{\text{iso}} < q_1^{\text{iso}}$ and the least eigenvalue corresponds to $\mathbf{m} = \mathbf{n}$. If $I_4 < 1$ and $F'' > 0$ then $q_1^{\text{fib}} < 0$ and $q_1^{\text{fib}} < q_2^{\text{fib}}$, which admits the possibility of $q_2^{\text{fib}} < 0$, so that (95) might vanish prior to vanishing of (93). This is also associated with $\mathbf{m} = \mathbf{n} = \hat{\mathbf{a}}$ and has the same interpretation. Similarly, if $F'' < 0$, in which case $q_2^{\text{fib}} < q_1^{\text{fib}} < 0$.

If $I_4 > 1$ then there is no loss of ellipticity if $F'' > 0$, as in Case (i), but if $F'' < 0$ then (95) may again vanish with the same interpretation as in Case (i).

At this point we emphasize that the examples of failure of ellipticity discussed in the above two cases do not necessarily correspond to a point at which ellipticity is *first* lost on a path of deformation from the reference configuration. Depending on the material model and the path of deformation, weak surfaces may be generated at an earlier point for which $\mathbf{n}$ is neither parallel to nor perpendicular to the deformed fiber direction $\mathbf{a}$. This can be appreciated by noting that after a little rearrangement $\det \mathbf{Q}$, as given by (72), may be expressed in the form

$$\det \mathbf{Q} = (q_1^{\text{iso}} + q_1^{\text{fib}})(q_2^{\text{iso}} + q_2^{\text{fib}}) + (q_2^{\text{iso}} - q_1^{\text{iso}})(q_2^{\text{fib}} - q_1^{\text{fib}})(\mathbf{n} \times \hat{\mathbf{a}})^2. \tag{97}$$

Cases (i) and (ii) above correspond to $\det \mathbf{Q} = 0$ with $\mathbf{n} = \hat{\mathbf{a}}$ and either $q_1^{\text{iso}} + q_1^{\text{fib}} = 0$ or $q_2^{\text{iso}} + q_2^{\text{fib}} = 0$. Clearly, $\det \mathbf{Q}$ could in principle vanish either in fiber extension or contraction on a path of deformation from the reference configuration with $q_1^{\text{iso}} + q_1^{\text{fib}} > 0$ and $q_2^{\text{iso}} + q_2^{\text{fib}} > 0$ if either $d > 0$ with $F'' < 0$ or $d < 0$ with $F'' > 0$ and $\mathbf{n} \cdot \mathbf{a} \neq 0$, $\mathbf{n} \times \mathbf{a} \neq \mathbf{0}$. In particular, it can vanish for small $(\mathbf{n} \cdot \mathbf{a})^2$, as illustrated in Section 3.1.2, Case (b). This corresponds to a weak surface close to parallel to the fiber direction (interpreted as fiber de-bonding in extension and fiber splitting in contraction).

For appropriate choices of material model and deformations prediction of a weak surface at (or close to) an angle of $\pi/4$ to the fiber direction is possible, but we do not pursue the details of this here. In compression such failure of ellipticity could be associated with the initiation of a shear band.

3.2. Coupled reinforcement

An alternative possible reinforcing model, which is the counterpart of that used in the incompressible theory (Merodio and Ogden, 2002), is given by

$$\widehat{W}(I_1, I_3, I_4) = W_{\text{iso}}(I_1, I_3) + W_{\text{fib}}(I_5), \tag{98}$$

where $I_5 = (I_1 - 1)I_4 - I_3$. Thus, while in (63) (I_1, I_3) and I_4 are decoupled, in (98) there is a coupling of (I_1, I_3) and I_4 through I_5. In Merodio and Ogden (2002) the notation $G(I_5) = W_{\text{fib}}(I_5)$ was used. For the present purposes we illustrate the typical characteristics that arise by using a modified form of I_5, denoted I_5^* and defined by

$$I_5^* = (I_1 - 1)I_4 - 1, \tag{99}$$

which couples I_4 with I_1 rather than with both I_1 and I_3. The algebra is much more involved if I_5 is used instead, but does not affect significantly the qualitative nature of the results, except that in extension loss of ellipticity can occur even if $G(I_5)$ is convex. We therefore replace (98) by

$$\widehat{W}(I_1, I_3, I_4) = W_{\text{iso}}(I_1, I_3) + W_{\text{fib}}(I_5^*) \tag{100}$$

and we use the notations

$$E(I_1, I_3) = W_{\text{iso}}(I_1, I_3), \quad G(I_5^*) = W_{\text{fib}}(I_5^*). \tag{101}$$

From (48) it can be shown that the contribution of the term G to the normal stress in the deformed fiber direction is $2J^{-1}I_4[\hat{\mathbf{a}} \cdot (\mathbf{B}\hat{\mathbf{a}}) + I_1 - 1]G'$, in which the coefficient of G' is positive. It is therefore appropriate to follow the pattern adopted in Section 2.4 and assume that G' satisfies (39) and (40).

On substitution of (98) into (53) we obtain

$$(\mathbf{Q}^{\text{iso}}\mathbf{m}) \cdot \mathbf{m} + (\mathbf{Q}^{\text{fib}}\mathbf{m}) \cdot \mathbf{m} > 0, \tag{102}$$

where $\mathbf{Q}^{\text{iso}}$ is as defined in Section 3.1.2 but $\mathbf{Q}^{\text{fib}}$ is now derived from G, such that

$$(\mathbf{Q}^{\text{fib}}\mathbf{m}) \cdot \mathbf{m} = 2G''[I_4\mathbf{m} \cdot (\mathbf{B}\mathbf{n}) + (I_1 - 1)(\mathbf{n} \cdot \mathbf{a})(\mathbf{m} \cdot \mathbf{a})]^2 + G'\left[I_4\mathbf{n} \cdot (\mathbf{B}\mathbf{n}) + (I_1 - 1)(\mathbf{n} \cdot \mathbf{a})^2\right] \tag{103}$$

from which it is clear that the coefficient of G' is strictly positive and that of G'' non-negative.

We first note the special case of (103) for which $\mathbf{n} \cdot \mathbf{a} = 0$. In contrast to the corresponding situation for F the terms in G in (103) do not vanish and reduce to

$$2G''[I_4\mathbf{m} \cdot (\mathbf{B}\mathbf{n})]^2 + G'I_4\mathbf{n} \cdot (\mathbf{B}\mathbf{n}). \tag{104}$$

In Section 3.1 it was noted that F does not admit a weak surface aligned with fiber direction. This is not the case for G. Recall from (39) that $G'(I_5^*) < 0$ for $I_5^* < 1$. If, for example, we take $\mathbf{m} = \hat{\mathbf{a}}$ and let $\mathbf{n}$ coincide with the principal axis of $\mathbf{B}$ corresponding to the stretch λ_1 then the left-hand side of (102) simplifies to $(E_1 + G'I_4)\lambda_1^2$. Since $E_1 > 0$ and $G'(1) = 0$ this expression is positive in the undeformed configuration but can vanish as I_5^* decreases from unity, at which point ellipticity is lost and the associated weak surface is aligned with the fiber direction. As in Merodio and Ogden (2002) we identify this failure of ellipticity with fiber splitting (Lee et al., 2000). If $I_5^* < 1$ we have $I_4 < 2/(\lambda_1^2 + \lambda_2^2)$. If, also, $\lambda_1^2 + \lambda_2^2 \geqslant 2$ (or $I_3 \geqslant 1$), which is the case for an *isochoric* deformation, then $I_4 < 1$ and the fiber is under contraction, but this is not so in general when $I_5^* < 1$, in contrast to the situation for an incompressible material. Similarly, if $I_4 > 1$ and $I_3 \geqslant 1$ then $I_5^* > 1$ follows.

If we now consider $\mathbf{n} = \hat{\mathbf{a}}$ and interchange the roles of $\mathbf{m}$ and $\mathbf{n}$ in the above paragraph (with $\mathbf{n}$ still corresponding to stretch λ_1) then the left-hand side of (102) becomes $E_1\lambda_1^2 + G'I_4(2\lambda_1^2 + \lambda_2^2)$. In this case ellipticity can fail as I_5^* decreases at a value closer to unity than for the above example. This corresponds to a weak surface normal to the fiber direction. If the fiber is under contraction this failure of ellipticity can correspond to fiber kinking, as for the $F(I_4)$ reinforcement.

Weak surfaces both aligned with and normal to the fiber direction may arise at the same deformation in special situations and hence fiber splitting and fiber kinking may occur simultaneously (Lee et al., 2000).

If the degree of anisotropy is sufficiently strong then the terms in G dominate the left-hand side of (102) and hence loss of ellipticity cannot be avoided under contraction if $I_5^* < 1$ is sufficiently small.

We now turn our attention to $I_5^* > 1$ so that $G' > 0$. It follows from (103) that a necessary condition for loss of ellipticity (if the base material is strongly elliptic) is $G''(I_5^*) < 0$. In this case the weak surface may be either (a) parallel to the fiber direction ($\mathbf{n} \cdot \mathbf{a} = 0$) or (b) normal to the fiber direction ($\mathbf{n} = \hat{\mathbf{a}}$). For (a) the appropriate failure mechanism is de-bonding, while in (b) it is matrix failure.

It can be shown, similarly to the situation described in Section 3.1.3, that if $G'' \geqslant 0$ then $G = \alpha g$ satisfies a nesting property with respect to the anisotropy parameter α. We also remark that if the isotropic base material loses ellipticity then overall ellipticity can fail either for $\mathbf{n} \cdot \mathbf{a} = 0$ or $\mathbf{n} = \hat{\mathbf{a}}$. With reference to (103), it can be seen that this can occur for $G'(I_5^*)$ and $G''(I_5^*)$ with appropriate signs.

3.2.1. A specific material model

On the same basis as the discussion in Section 3.1.4 we consider an isotropic base material as given in (82) but with $F(I_4)$ replaced by $G(I_5^*)$. We again denote by $(q_1^{\rm fib}, q_2^{\rm fib})$ the eigenvalues of the acoustic tensor $\mathbf{Q}^{\rm fib}$ associated with the reinforcing model. Then it can be shown that the eigenvalues are given by

$$q_1^{\rm fib} = \mathbf{n} \cdot \boldsymbol{v} G', \quad q_2^{\rm fib} = q_1^{\rm fib} + 2v^2 G'', \tag{105}$$

where

$$v = |\,\boldsymbol{v}\,|, \quad \boldsymbol{v} = I_4 \mathbf{B}\mathbf{n} + (I_1 - 1)(\mathbf{n} \cdot \mathbf{a})\mathbf{a} \tag{106}$$

and we note that $\mathbf{n} \cdot \boldsymbol{v} > 0$ for all $\mathbf{n} \neq \mathbf{0}$. The associated eigenvectors satisfy, respectively, $\mathbf{m} \cdot \boldsymbol{v} = 0$ and $\mathbf{m} \times \boldsymbol{v} = \mathbf{0}$.

Analogously to (97) we calculate

$$\det \mathbf{Q} = (q_1^{\rm iso} + q_1^{\rm fib})(q_2^{\rm iso} + q_2^{\rm fib}) + (q_2^{\rm iso} - q_1^{\rm iso})(q_2^{\rm fib} - q_1^{\rm fib})(\mathbf{n} \times \hat{\boldsymbol{v}})^2, \tag{107}$$

where $\hat{\boldsymbol{v}} = \boldsymbol{v}/v$. The combinations $(q_1^{\rm iso}, q_2^{\rm fib})$ and $(q_2^{\rm iso}, q_1^{\rm fib})$ are not admissible since they both entail $v = 0$, which is a contradiction. The combinations $(q_1^{\rm iso}, q_1^{\rm fib})$ and $(q_2^{\rm iso}, q_2^{\rm fib})$ each lead to $\boldsymbol{v} = v\mathbf{n}$, and we then have

$$q_1^{\rm iso} + q_1^{\rm fib} = b_1 n_1^2 + b_2 n_2^2 + vG', \tag{108}$$

$$q_2^{\rm iso} + q_2^{\rm fib} = d + b_1 n_1^2 + b_2 n_2^2 + vG' + 2v^2 G''. \tag{109}$$

Finally, we illustrate these results by using the same form for $E(I_1, I_3)$ and the same deformation as was used in Section 3.1.4 with $I_3 = 1$ and the standard model for G, written

$$G(I_5^*) = \tfrac{1}{2}\alpha(I_5^* - 1)^2. \tag{110}$$

For the considered specialization we have $I_5^* = \lambda^4$.

The results are shown in Fig. 2, in which $\det \mathbf{Q} = 0$ is plotted in the (λ^2, n_1^2) plane for the same values of the material constants $\bar{\alpha}$ and δ as were used in Fig. 1. For $\bar{\alpha} = 5$ the result is similar qualitatively to those shown in Fig. 1, but in this case, as $\bar{\alpha}$ increases, two separate branches emerge. For each of the two larger values of $\bar{\alpha}$ the possibility of loss of ellipticity on a weak surface parallel to the fibers is evident (corresponding to $n_1 = 0$) in addition to that normal to the fiber direction ($n_1 = 1$). However, for the considered model loss of ellipticity occurs first as λ decreases from 1 on a weak surface normal to the fiber direction, although as $\bar{\alpha}$ increases the values of λ become very close for the two modes of ellipticity loss.

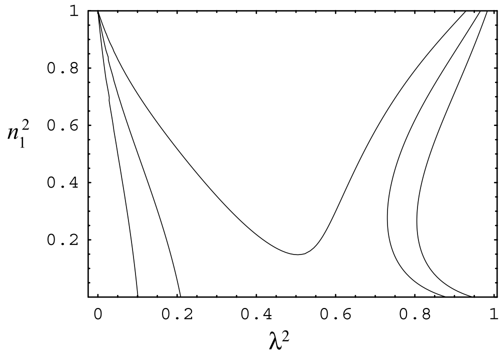

Fig. 2. Plot of the value of n_1^2 against λ^2 in compression corresponding to loss of ellipticity using the reinforcing model (110) and dimensionless material parameters $\delta = 40$ and $\bar{\alpha} = 5$, 10, 20.

4. Discussion and summary

This analysis has been motivated by instability phenomena in fiber-reinforced composite materials and has focused on failure prediction on the basis of loss of ellipticity of the considered elastic material. The materials considered are isotropic base materials augmented by a function that accounts for the existence of fiber reinforcement (the reinforcing model). A detailed analysis of the ellipticity status of the I_4 reinforcing model has been given. In particular, in Section 3.1 simple conditions that guarantee the ellipticity of the I_4 reinforcing model have been derived. As in the case of an incompressible material (Merodio and Ogden, 2002) it was found that loss of ellipticity (and hence fiber failure) is to be expected under fiber contraction. Failure may also occur under fiber extension if the reinforcing model is non-convex. In Section 3.2, an alternative reinforcing model has been considered briefly and its effect on the loss of ellipticity has been illustrated in some simple cases. We have indicated how the breakdown of ellipticity might be related to different fiber failure mechanisms—kinking and splitting in compression and de-bonding or matrix failure in tension.

Other possible models can be considered and may extend the range of possible failure mechanisms. For example, in compression, when the weak surface is neither close to alignment with nor normal to the fiber direction (and, in particular, when it bisects these directions) then it may be considered as associated with initiation of a shear band. Note, however, that elasticity theory per se cannot predict shear band thickness. The analysis of shear band development requires an inelastic theory involving a length scale. A similar comment applies to kink bands. We have examined only instabilities associated with loss of ellipticity in a homogeneous material homogeneously deformed, so that boundary conditions are not involved. We have not considered other types of instability such as buckling, which, under appropriate boundary conditions, may be initiated prior to loss of ellipticity.

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